MATH 119 Chapter 3

0.1 Definition 1: Informal definition of limit

The function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal, a.

The following notation is used

$$\lim_{x \to a} f(x) = l$$

For a mathematical correct and complete definition see any Calculus book used in Math 112).

0.2 Existence of limits

The $\lim_{x \to a} f(x)$ does not exist if

a) $\lim_{x \to a^-} f(x) = l$, $\lim_{x \to a^+} f(x) = m$, and $l \neq m$.

b) $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

0.3 Properties of Limits

If

$$\lim_{x \to a} f(x) = l, \quad \lim_{x \to a} g(x) = m,$$

and c is a real constant, then

a)

$$\lim_{x \to a} c f(x) = c\, l.$$

b)

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l \pm m.$$

c)

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = l \cdot m.$$

d)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}, \quad \text{if} \quad \lim_{x \to a} g(x) \neq 0.$$

e)

$$\lim_{x \to a} [f(x)]^c = [\lim_{x \to a} f(x)]^c = l^c.$$

f)

$$\lim_{x \to a} c^{f(x)} = c^{\lim_{x \to a} f(x)} = c^l.$$

h)

$$\lim_{x \to a} [\ln f(x)] = \ln[\lim_{x \to a} f(x)] = \ln(l), \quad \text{if} \quad l > 0.$$

0.4 Definition of Continuity

A function f is continuous at $x = c$ if the following conditions are satisfied:

a) $f(c)$ is defined

b) $\lim_{x \to c} f(x)$ exists.

c) $\lim_{x \to c} f(x) = f(c)$.

0.5 Definition of Continuity in an open interval

A function f is continuous in an open interval (a, b) if f is continuous for all x in (a, b).

0.6 Definition of Continuity in a closed interval

A function f is continuous in a closed interval $[a, b]$ if the following conditions are satisfied:

a) f is continuous in (a, b).

b) $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

0.7 Intermediate Value Theorem

If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there is some x in $[a, b]$ such that $f(x) = c$.

0.8 Definition of Average Rate of Change

The average rate of change of a function f with respect to x as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}$$

0.9 Definition of Instantaneous Rate of Change

The instantaneous rate of change of a function f with respect to x at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

0.10 Definition of Tangent Line to the Graph of a Function

The tangent line to the graph of a function f at the point $(a, f(a))$ is the line passing through $(a, f(a))$ with slope

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

0.11 Definition of Differentiability

The function f is differentiable at a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{exists.}$$

In this case the limit is denoted by $f'(a)$ or

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

and is called the derivative of f at a.

0.12 Definition of Derivative Function

If the function f is differentiable at every x in the interval (a, b), then the function defined as $f'(x)$ for every x in (a, b) is called the derivative function of f in (a, b) and is denoted by the symbol f'.

0.13 Definition of Tangent Line to the Graph of a Function (Revisited)

For a function f which is differentiable at a, the tangent line to the graph of f at the point $(a, f(a))$ is the line

$$y = f'(a)(x - a) + f(a).$$

Remark: *If the function f is not differentiable at a, then there is not tangent line to the graph of f at $(a, f(a))$*